DIRECT DERIVATION OF THE ORDINARY CANONICAL SYSTEM OF ELLIPTIC ELEMENTS EMPLOYED IN THE PROBLEM OF THREE BODIES.

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The canonical equations of motion may be written

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \tag{1}$$

in which

$$H = \Sigma p_i \frac{dq_i}{dt} - T - F \tag{2}$$

is a constant, and

$$p_i = \frac{\partial T}{\partial \frac{dq_i}{dt}}. \tag{3}$$

T is a function of the q_i's and $\frac{dq_i}{dt}$'s but H is expressed as a function of the p_i's and q_i's.

We will consider the problem of three bodies in which

$$T \equiv \tfrac{1}{2}\left[\frac{dr}{dt}\right]^2 + \tfrac{1}{2} r^2 \left[\frac{dw}{dt}\right]^2, \tag{4}$$

$$F \equiv \frac{\mu}{r} + R_1, \tag{5}$$

$$H = \tfrac{1}{2}\left[\frac{dr}{dt}\right]^2 + \tfrac{1}{2} r^2 \left[\frac{dw}{dt}\right]^2 - \frac{\mu}{r} - R_1, \tag{6}$$

where r, w are the radius vector and longitude in orbit of the disturbed body, R_1 is a function of the coordinates of both the disturbed and disturbing bodies, and μ is a constant.

As q_i's we will select

$q_1 \equiv$ the mean anomaly,

$q_2 \equiv$ the angular distance of the perihelion from the node,

$q_3 \equiv$ the longitude of the node counted from a fixed point in the fundamental plane; in each case reference being had to the instantaneous Keplerian ellipse. It remains to find p_1, p_2, p_3.

dq_1/dt is independent of dq_2/dt and dq_3/dt; it is also independent of the form of the orbit. In deriving p_1, therefore, by means of (3), we may write T_0 in the place of T, the former being obtained on the assumption that the orbit

is a circle whose radius is the semi-major axis, a, of the instantaneous ellipse. In the instantaneous ellipse, instead of (6) we have

$$\tfrac{1}{2}\left[\frac{dr}{dt}\right]^2 + \tfrac{1}{2}r^2\left[\frac{dw}{dt}\right]^2 - \frac{\mu}{r} + \frac{\mu}{2a} = 0\,; \tag{7}$$

In the corresponding circle $dr/dt = 0$, $r = a$, $dw/dt = dq_1/dt$; whence

$$T_0 = \tfrac{1}{2}a^2\left[\frac{dq_1}{dt}\right]^2, \tag{8}$$

and

$$p_1 = \frac{\partial T_0}{\partial \dfrac{dq_1}{dt}} = a^2\frac{dq_1}{dt}\,. \tag{9}$$

Also, from (7), on the same assumption,

$$\tfrac{1}{2}a^2\left[\frac{dq_1}{dt}\right]^2 - \frac{\mu}{2a} = 0\,, \tag{10}$$

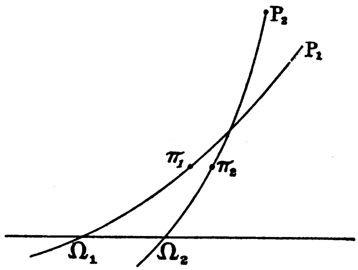
or

$$\frac{dq_1}{dt} = \frac{\sqrt{\mu}}{a^{\frac{3}{2}}}\,. \tag{11}$$

Combining (9) and (11), we have

$$p_1 = \sqrt{\mu a}\,. \tag{a}$$

A reference to the accompanying figure (in which P_1, P_2 are successive positions of the disturbed body on the celestial sphere, and Ω_1, Ω_2; π_1, π_2 are



the corresponding positions of the instantaneous node and perihelion) will show that

$$\frac{dw}{dt} = \frac{dv}{dt} + \frac{dq_2}{dt} + \frac{dq_3}{dt}\cos i\,, \tag{12}$$

where v is the instantaneous true anomaly, and i is the inclination of the

instantaneous orbit to the fundamental plane; and since dq_2/dt is independent of dr/dt, dv/dt, and dq_3/dt, equations (3), (4), and (12) give

$$p_2 = \frac{\partial T}{\partial \frac{dq_2}{dt}} = r^2 \frac{dw}{dt}\,. \tag{13}$$

Substituting in (7),

$$\tfrac{1}{2}\left[\frac{dr}{dt}\right]^2 + \tfrac{1}{2}\frac{p_2^{\,2}}{r^2} - \frac{\mu}{r} + \frac{\mu}{2a} = 0\,. \tag{14}$$

This equation holds true for all points of the instantaneous ellipse. For the maximum and minimum values of r, since for these values $dr/dt = 0$, we have

$$\tfrac{1}{2}\frac{p_2^{\,2}}{r^2} - \frac{\mu}{r} + \frac{\mu}{2a} = 0\,,$$

or

$$r^2 - 2ar + \frac{a}{\mu}p_2^{\,2} = 0\,. \tag{15}$$

Calling the roots of this equation $r_1 = a(1 - e)$ and $r_2 = a(1 + e)$, as given by the properties of the ellipse, the last term of (15) gives

$$r_1 r_2 = \frac{a}{\mu}p_2^{\,2} = a^2(1 - e^2)\,,$$

or

$$p_2 = \sqrt{\mu a(1 - e^2)} = p_1\sqrt{1 - e^2}\,, \tag{b}$$

in which, of course, e is the eccentricity of the instantaneous ellipse.

Finally, since dq_3/dt is independent of dr/dt, dv/dt, dq_2/dt, equations (3), (4), and (12) give

$$p_3 = \frac{\partial T}{\partial \frac{dq_3}{dt}} = r^2\frac{dw}{dt}\cos i = p_2 \cos i\,. \tag{c}$$

A comparison of (7) with (6) gives

$$H = -\frac{\mu}{2a} - R_1 = -\frac{\mu^2}{2p_1^{\,2}} - R_1 = -R\,,$$

in which R is the expression used by Delaunay in his Théorie de la Lune.

The derivation of the ordinary canonical system here given has the following apparent advantages: 1. The use of Hamilton's principal function is avoided; 2. The argument forming Chap. 36 of Jacobi's Vorlesungen über Dynamik, or that forming Vol. I, Art. 59, of Tisserand's Mécanique Céleste, is rendered unnecessary in applying the system to the theory of perturbations; 3. The reason for the addition of the term $\mu^2/(2p_1^{\,2})$ to the perturbative function is shown, without a special investigation, such as that given in Vol. I, Art. 5, of Delaunay's Théorie de la Lune.